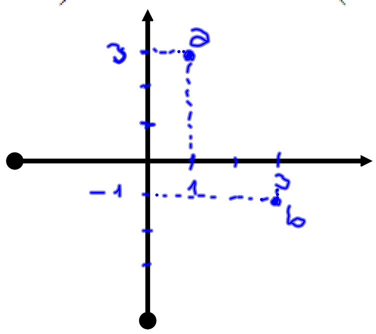
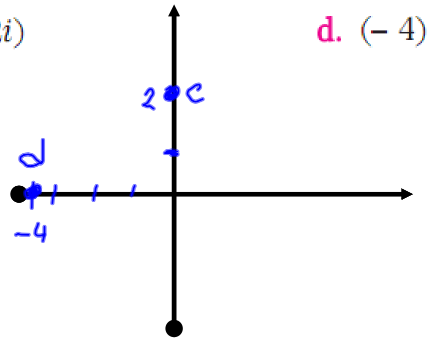
A. Complex Numbers

1. Locate the points that represent the complex numbers given below in the complex plane.

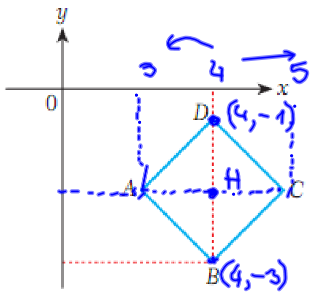
a. $(1 + 3i)$ b. $(3 - i)$



c. $(2i)$ d. (-4)



2. In the figure, $ABCD$ is a square. If the complex numbers $z_1 = 4 - 3i$ and $z_2 = 4 - i$ are represented by the points B and D, find the complex numbers that are represented by the points A and C.



|DB| = |-1-(-3)| = 2

midpoint of B and D is (4,?)

|DH| = 1 unit
|HB| = 1 unit
so H(4,-2)

then so A(? ; -2) C(? ; -2)

|AH| = 1 unit = |HC|

A(3,-2)
C(5,-2

B. Conjugate of a Complex Number

3. Find the conjugates of following complex numbers.

a. $3 \overset{z}{+} 2i$

$\bar{z} = 3 - 2i$

b. $4 - i$

$\bar{z} = 4 + i$

c. $2i - 1$

$\bar{z} = -2i - 1$

d. $\sqrt{3} - 1 - i$

$\bar{z} = \sqrt{3} - 1 + i$

4. For the given complex number $z = 1 + 4i$, represent z, $\bar{z}$, $-z$, and $-\bar{z}$ in a complex plane.

$z = 1 + 4i$

$\bar{z} = 1 - 4i$

$-z = -(1 + 4i) = -1 - 4i$

$-\bar{z} = -1 + 4i$

C. Basic Operations in Complex Numbers

5. $z_1 = -2 + 3i$ and $z_2 = 3 - 2i$ are given. Evaluate the following:

a. $z_1 + z_2 = 1 + i$

b. $z_2 - 4z_1 = 3 - 2i - (-8 + 12i)$
$= 11 - 14i$

c. $z_2 - z_1 = 5 - 5i$

d. $z_1 \cdot z_2 =$
$= (-2+3i)\cdot(3-2i) =$
$= -\cancel{6} + 4i + 9i - \cancel{6i^2} = 13i$

e. $\frac{z_1}{z_2} = \frac{(-2+3i)}{(3-2i)} \cdot \frac{3+2i}{3+2i}$

f. z_2^{-1}

$= \frac{-6-4i+9i-6}{9+4} = \frac{-12+5i}{13} = -\frac{12}{13} + \frac{5}{13}i$

6. Find the real numbers x and y for each equation below.

a. $(1 - 2i)x + (1 + 2i)y = 1 + i$

$x - 2xi + y + 2yi = 1 + i$

$x + y + (-2x + 2y)i = 1 + i$

$x + y = 1 \quad / \cdot 2$
$-2x + 2y = 1$

$2x + 2y = 2$
$-2x + 2y = 1$

$4y = 3$
$\boxed{y = \frac{3}{4}}$

$x + \frac{3}{4} = 1$
$x = 1 - \frac{3}{4} = \frac{1}{4}$
$\boxed{x = \frac{1}{4}}$

b. $(2 + i)x - (2 - i)y = x - y + 2i$

⋮

7. Write each of the following complex numbers in standard form.

a. $(1+i)^3-(1-i)^3$

$= 1+3\cdot 1^2\cdot i+3\cdot 1\cdot i^2+i^3-(1-3\cdot 1\cdot i+3i^2-i^3)$

$= -2+2i-(-2-2i) = -\cancel{2}+2i+\cancel{2}+2i$

$= 4i$

b. $\left(\frac{1-i}{1+i}\right)^{1995} = \left(\frac{1-2i-1}{1+1}\right)^{1995}$

$(1-i)$

$= (-i)^{1995} = -i^3$

$= i$

c. $\frac{\overbrace{(1+i)^3}^{-2+2i}}{1-i}-\frac{\overbrace{(1-i)^3}^{-2-2i}}{1+i}$

$= \frac{-2\cancel{(1-i)}}{\cancel{1-i}} - \frac{-2\cancel{(1+i)}}{\cancel{1+i}}$

$= \cancel{-2+2} = 0$

d. $\left(\frac{1-\sqrt{3}i}{1+\sqrt{3}i}\right)^2$

8. Find the complex numbers z satisfying the following equalities.

a. $z-3+8i = \bar{z}\cdot(1+i)$

$\cancel{x}+yi-3+8i = (x-yi)\cdot(1+i) = \cancel{x}+xi-yi+y$

$\underbrace{-3+(y+8)i} = y+(x-y)i$

$y+8 = x-y$

$2\overset{-3}{y}+8 = x$

$\boxed{x=2}$

b. $z\cdot(2-i)+3(\bar{z}-4) = 1-i$

9. Given $\overline{z} = \dfrac{3-4i}{1+2i}$. Find z^{-1}.

$$\overline{z} = \frac{3-4i}{1+2i} \cdot \frac{1-2i}{1-2i} = \frac{3-6i-4i+8}{1+4} = \frac{11}{5} - 2i$$

$$z = \frac{11}{5} + 2i = \frac{11+10i}{5}$$

$$z^{-1} = \frac{5}{11+10i} \cdot \frac{11-10i}{11-10i} = \frac{55-50i}{121+100} = \frac{55}{221} - \frac{50}{221}i$$

10. Find $Im(z)$, if $z = \dfrac{\sqrt{2}+i}{\sqrt{2}-i} + i \cdot \dfrac{\sqrt{2}-i}{\sqrt{2}+i}$.

$$(\sqrt{2}+i) \qquad (\sqrt{2}-i)$$

$$z = \frac{2+2\sqrt{2}i \cancel{-1}}{3} + \frac{\sqrt{2}i \cancel{+1}}{3} = \frac{2}{3} + \frac{\cancel{3}\sqrt{2}i}{\cancel{3}} = \frac{2}{3} + \sqrt{2}i$$

D. Modulus of a Complex Number

11. Find the modulus of each of the following complex numbers and represent them in a complex plane.

a. $z = -4 - 3i$ **b.** $v = -5 + 12i$

$|z| = \sqrt{(-4)^2 + (-3)^2}$

$= \sqrt{16 + 9} = 5$

12. Find $|z|$ for each of the following complex numbers.

a. $z = \dfrac{(1+i)\cdot(1-2i)^2}{(-3+i)\cdot(2+i)}$ **b.** $z = \dfrac{1}{\sqrt{3} - i}$

$|z| = \left|\dfrac{(1+i)\cdot(1-2i)^2}{(-3+i)\cdot(2+i)}\right|$

$= \dfrac{|1+i|\cdot|1-2i|^2}{|-3+i|\cdot|2+i|} = \dfrac{\sqrt{2}\cdot(\sqrt{5})^2}{\sqrt{10}\cdot\sqrt{5}}$

$= \dfrac{\sqrt{2}\cdot 5}{\sqrt{2}\cdot\sqrt{5}\cdot\sqrt{5}} = 1$

$|z| = \dfrac{|1|}{|\sqrt{3} - i|} = \dfrac{1}{\sqrt{4}} = \dfrac{1}{2}$

13. Given $z_1 = x + i$, $z_2 = 2 - 2i$ and $\left|\overline{z_1 + z_2}\right| = \sqrt{10}$.
Find x.

$z_1 + z_2 = 2 + x - i$

$\overline{z_1 + z_2} = 2 + x + i$

$\left|\overline{z_1 + z_2}\right| = \sqrt{(2+x)^2 + 1^2} = \sqrt{10}$

$(2+x)^2 + 1 = 10$

$(2+x)^2 = 9$

$2 + x = \mp 3$

$x_1 = 1$ $\qquad$ s.s. $= \{1, -5\}$

$x_2 = -5$

14. Find $|z^{-1}|$, if $z = (1 - i) \cdot (2 + 3i)$.

$|z^{-1}| = |z|^{-1}$

$|z| = |(1-i) \cdot (2+3i)| = |1-i| \cdot |2+3i| = \sqrt{1^2 + (-1)^2} \cdot \sqrt{2^2 + 3^2}$

$|z| = \sqrt{2} \cdot \sqrt{13} = \sqrt{26}$

$|z|^{-1} = \dfrac{1}{\sqrt{26}}$

15. For $z = 3 + i$, find $|z - 2 + i| + |z + 1 - 3i|$.

$= |3+i-2+i| + |3+i+1-3i| =$

$= |1+2i| + |4-2i|$

$= \sqrt{1^2+2^2} + \sqrt{4^2+(-2)^2}$

$= \sqrt{5} + \sqrt{20} = \sqrt{5} + 2\sqrt{5} = 3\sqrt{5}$

16. Find the complex numbers z satisfying the following equalities. Let $z = a+bi$

a. $|z| + z = 6 - 2i$

$|a+bi| + a + bi = 6 - 2i$

$i^2 = -1$

$\sqrt{a^2+b^2} = (6-a) - (2+b)i$

$a^2+b^2 = (6-a)^2 \underbrace{- 2\cdot(6-a)\cdot(2+b)i}_{Im} - (2+b)^2$

$\cancel{a^2} + b^2 = (6-a)^2 - (2+b)^2 = 36 - 12a + \cancel{a^2} - 4 - 4b - b^2$

$\boxed{2b^2 = 32 - 12a - 4b}$

$-2\cdot(6-a)\cdot(2+b) = 0$

$\boxed{a = 6,\ b = -2}$

for $a = 6$, $2b^2 = 32 - 72 - 4b \quad /:2$

$b^2 + 2b + 20 = 0$

$\Delta < 0$

no real root for b ($b \in \mathbb{R}$)

$\Rightarrow$

for $b = -2$

$2\cdot(-2)^2 = 32 - 12a - 4\cdot(-2)$

$\cancel{8} = 32 - 12a + \cancel{8}$

$a = \frac{8}{3}$

$\boxed{z = \frac{8}{3} - 2i}$

b. $z - |\overline{z}| = -1 + 2i$

Let z be $a+bi$

$a+bi - |\overline{a+bi}| = -1+2i$

$a+bi - |a-bi| = -1+2i$

$a+bi - \sqrt{a^2+(-b)^2} = -1+2i$

$a+1+(b-2)i = \sqrt{a^2+b^2} + 0i$

$b-2=0$

$b=2$

$(a+1)^2 = (\sqrt{a^2+b^2})^2$

$\cancel{a^2}+2a+1 = \cancel{a^2}+4$

$2a = 3$

$a = \frac{3}{2}$

$$\boxed{z = \frac{3}{2} + 2i}$$

E. Distance Between two Complex Numbers

17. Find the distance between each pair of complex numbers given beow.

a. $z_1 = 2 - 3i$ and $z_2 = -1 + i$

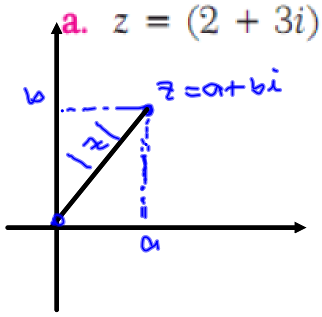
$|z_1 - z_2| = |z_2 - z_1|$ represents distance between z_1 and z_2

$= |2-3i-(-1+i)| = |2+1-3i-i| = |3-4i|$

$= \sqrt{3^2+(-4)^2} = \sqrt{9+16} = \sqrt{25} = 5$ units

18. Find the distance of each complex number to the origin given below.

a. $z = (2+3i)^2 \cdot (4+3i)$



$|z| = |(2+3i)^2 \cdot (4+3i)| = |2+3i|^2 \cdot |4+3i|$

$|z| = \left(\sqrt{2^2+3^2}\right)^2 \cdot \sqrt{4^2+3^2}$

$= 13 \cdot 5 = 65$

b. $|u| = \left| \dfrac{(3-2i)\cdot\sqrt{13-3\sqrt{3}i}}{2+3i} \right|$

$|u| = \dfrac{|3-2i| \cdot \sqrt{|13-3\sqrt{3}i|}}{|2+3i|}$

be careful

$|z|^n = |z^n|$

$|z^{\frac{1}{2}}| = |z|^{\frac{1}{2}} = \sqrt{|z|}$

169
27
$196 = 14^2$

$= \dfrac{\cancel{\sqrt{13}} \cdot \sqrt{\sqrt{14^2}}}{\cancel{\sqrt{13}}} = \sqrt{14}$ //

F. Lines and Circles in the Complex Plane

19 . Find and geometrically represent the locus of the complex numbers z satisfying each of the following equation.

a. $|z-2| = |z+1|$

let z=x+yi

$|x+yi-2| = |x+yi+1|$

$|x-2+yi| = |x+1+yi|$

$\cancel{\sqrt{}}\,(x-2)^2+y^2 = \cancel{\sqrt{}}\,(x+1)^2+y^2$

$\cancel{x^2}-4x+4 = \cancel{x^2}+2x+1$

$3 = 6x$

$1 = 2x$

$\boxed{0 = 2x-1}$ → equ. of a line

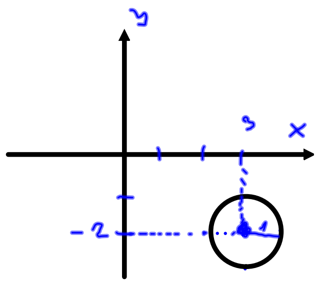
d. $|z-3+2i| = 1$ ($z \to x+yi$)

$|x-3+(y+2)i| = 1$

$\sqrt{(x-3)^2+(y+2)^2} = 1$

$(x-3)^2+(y+2)^2 = 1^2$

↳ circle



c. $|z| = 2$

b. $|z-1+2i| = |\overline{z}+2-i|$

e. $|\overline{z}-i| = 4$

⋮

20 . Find the complex number z satisfying each of the given systems below.

a. $\begin{cases} |z-i| = |z-1| \ldots ① \\ |z+2+i| = |\bar{z}-3| \ldots ② \end{cases}$

① $|x+(y-1)i| = |x-1+yi|$

$\sqrt{x^2+(y-1)^2} = \sqrt{(x-1)^2+y^2}$

$\cancel{x^2}+\cancel{y^2}-2y+\cancel{1} = \cancel{x^2}-2x+\cancel{1}+\cancel{y^2}$

$x-y=0$

$x=y$

② $|x+2+(y+1)i| = |x-3-yi|$

$\sqrt{(x+2)^2+(y+1)^2} = \sqrt{(x-3)^2+(-y)^2}$

$\cancel{x^2}+4x+4+\cancel{y^2}+2y+1 = \cancel{x^2}-6x+9+\cancel{y^2}$

$10x+2y=4 \quad /:2$

$5x+y=2$

$6x=2 \Rightarrow \boxed{x=y}$

$x=\frac{1}{3}$

$y=\frac{1}{3}$

$\boxed{z=\frac{1}{3}+\frac{1}{3}i}$

b. $\begin{cases} |2i-z| = |z+1-i| \\ |\bar{z}-i| = |z-4+2i| \end{cases}$

$\vdots$

21 . Represent the image of the following inequalities in the complex plane, where $z = x + yi$.

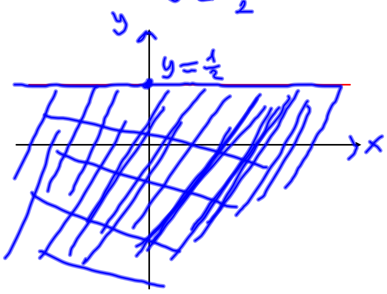
a. $|z| \leq |z-i|$

$\sqrt{x^2+y^2} \leq \sqrt{x^2+(y-1)^2}$

$x^2+y^2 \leq x^2+y^2-2y+1$

$0 \leq -2y+1$

$y \leq \frac{1}{2}$



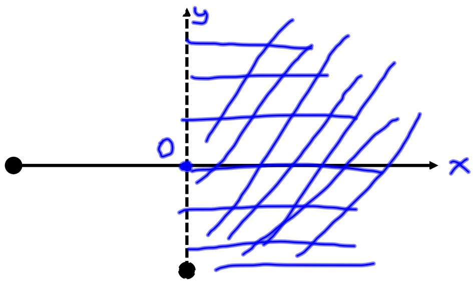
b. $|z+1| > |z-1|$

$\sqrt{(x+1)^2+y^2} > \sqrt{(x-1)^2+y^2}$

$x^2+2x+1+y^2 > x^2-2x+1+y^2$

$4x > 0$

$x > 0$

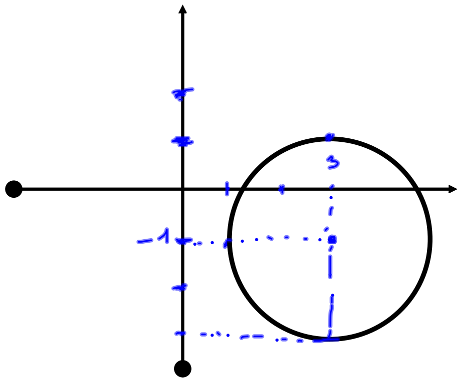


c. $|z - 4 - 4i| \geq 2$

d. $|z - 2i| < 3$

e. $2 \leq |z| < 4$

22 . Find the equation of the circle centered at the point corresponding to the complex number $3 - i$ with a radius 2 cm.



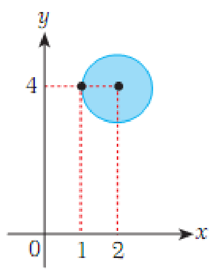
$(x-3)^2 + (y+1)^2 = 2^2$ ...... ①

$|z - (3-i)| = 2$

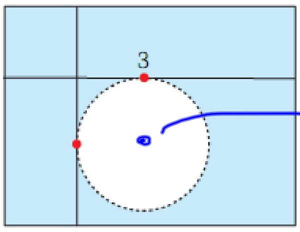
$|z - 3 + i| = 2$ ...... ②

23 . Define each of the following shaded regions given in the complex plane by means of complex numbers z.

a.



b.



$c(3,-3)$

$r=3$

a. $(x-2)^2+(y-4)^2=1^2$

$|z-2-4i|=1$

$\boxed{|z-2-4i|\leqslant 1}$

b. $(x-3)^2+(y+3)^2=3^2$

$|z-3+3i|=3$

$\boxed{|z-3+3i|>3}$

24 . Write the quadratic equation with real coefficents if one of the roots is given below.

a. $x_1 = 2 + 3i$

conjugate

$x_2 = 2-3i$

b. $x_1 = \sqrt{5} - i$

$x_2 = \sqrt{5} + i$

$x_1+x_2=4$

$x_1 \cdot x_2 = 2^2+3^2=13$

$\boxed{x^2-(x_1+x_2)x+x_1\cdot x_2=0}$

$\boxed{x^2-4x+13=0}$

c. $x_1 = \sqrt{3}\, i$

$x_2 = -\sqrt{3} i$

d. $x_1 = \sqrt{2} + \sqrt{2}\, i$

$x_2 = \sqrt{2} - \sqrt{2} i$

$\vdots$

$x_1 + x_2 = 0$

$x_1 \cdot x_2 = 3$

$x^2 - (x_1 + x_2)x + x_1 \cdot x_2 = 0$

$x^2 - 0.x + 3 = 0$

$\boxed{x^2 + 3 = 0}$